On the limiting form of the Kalmanbucy filter as measurement noise tends to zero (Part I,Finite Dimension)

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Rapports de Recherche

N° 226

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Juillet 1983

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RESUME

On étudie dans ce papier le comportement asymptotique du filtre de Kalman lorsque le bruit sur l'observation tend vers zéro. On montre que sous des hypothèses assez générales la forme limite du filtre correspond à un observateur de Luenberger d'un système déterministe.

ABSTRACT

The purpose of this paper is to show that, under general circumstances, the Kalman-Bucy filter approaches a meaningful limit as measurement noise tends to zero. Moreover, if it shown that the limiting form corresponds to the so-called Luenberger observer in deterministic systems under suitable assumptions.

C O N T E N T S

1 - INTRODUCTION

Let us consider the system

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t) + D\xi(t) \tag{1}$$

$$x(o) = x_o \qquad E[x_o] = \bar{x}_o$$

$$y(t) = Cx(t) + \eta(t) \tag{2}$$

where $\xi(t)$ and $\eta(t)$ are independent Gaussian white noise processes such that

$$E\,[\xi(t)\ \xi'(\tau)] = Q\delta(t-\tau) \tag{3}$$

$$E\,[\eta(t)\ \eta'(\tau)] = R\delta(t-\tau) \tag{4}$$

Then the optimum minimum variance estimator $\hat{x}(t) = E\,[x(t) \mid y(\tau) \quad 0 \le \tau \le t]$ is given by the well-known Kalman-Bucy filter [8] :

$$\text{(5)} \qquad \frac{d}{dt}\,\hat{x}(t) = A\hat{x}(t) + By(t) + P(t)\,C'R^{-1}\,[y(t) - C\hat{x}(t))]$$

with $P(t) = E\,[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))']$ being the solution to the matrix Ricatti equation

$$\text{(6)} \qquad \frac{d}{dt}\,P(t) = AP(t) + P(t)A' - P(t)C'R^{-1}\,CP(t) + DQD'$$

$$\text{(6)} \qquad P(o) = \Pi_o = E\,[(x_o - \bar{x}_o)(x_o - \bar{x}_o)']$$

under the assumption that the matrix R is positive definite.

If R is singular, however, the problem is much more complicated. One difficulty is that the gain $PC'R^{-1}$ of the filter becomes very large as the covariance matrix R of the noise η becomes small. In the case where R = 0, the nonsingularity of the matrix CDQD'C' means that the new measurement vector $\frac{d}{dt}\,y(t)$ will contain a non-singular white noise component [2], [6]. Therefore, a standard Kalman-Bucy filter may be derived by using $\frac{d}{dt}\,y(t)$ instead of $y(t)$, and the need for the differentiation can be avoided by defining new filter state $Z_d(t) = \hat{x}(t) - P(t)C'R^{-1}y(t)$, [6].

This works is very closely related to the results of FRIEDLAND [2], MOYLAN [6], and KWANTNY [3]. Especially, FRIEDLAND [2], MOYLAN [6] proved that a known class of minimal order observers can be identified as limiting forms of Kalman-Bucy filters for the new filter state $Z_d(t)$ with vanshing observation noise under the assumption that CDQD'C' is nonsingular.

The objective of this paper is to study the limiting form of Kalman-Bucy filter as R tends to zero under general circumstances, i.e. without assuming the regularity of CDQD'C'.

2 - SERIES EXPANSION

For the simplicity we consider the time invariant case, that is, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{n\times \ell}$, $Q \in \mathbb{R}^{\ell\times \ell}$, $R \in \mathbb{R}^{r\times r}$. In this case, provided that the pair (C,A) is detectable, and the 2n × 2n Hamiltonian matrix H defined by

$$H = \begin{bmatrix} A' & -C'R^{-1}C \\ -CDQD'C' & -A \end{bmatrix}$$

has no imaginary axis eigenvalues, P(t) tends to a solution of the algebraic Riccati equation (ARE)

$$(7) \qquad AP + PA' - PC'R^{-1}CP + GG' = 0$$

with $DQD' = GG'$ under suitable initial conditions, and $A - PC'C$ is stable [1]. Moreover, if (A,G) is stabilizable, the ARE (7) has an unique positive definite solution [7].

Let

$$(8) \qquad R = \varepsilon^2 I$$

then the ARE (7) becomes

$$(9) \qquad AP_\varepsilon + P_\varepsilon A' - \frac{1}{\varepsilon^2} P_\varepsilon C'CP_\varepsilon + GG' = 0$$

Suppose P has the series expansion of the form such that

$$(10) \qquad P_\varepsilon = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots$$

By substituting (10) into (9) and equating matrix coefficients of same powers of ε, we have

(11a) $CP_0 = 0$

(11b) $AP_0 + P_0A' - P_1C'CP_1 + GG' = 0$

(11c) $AP_1 + P_1A' - P_2C'CP_1 - P_1C'CP_2 = 0$

(11d) $AP_2 + P_2A' - P_2C'CP_2 - P_1C'CP_3 - P_3C'CP_1 = 0$

....

Pre- and postumultiplying (11b) by C and C', it follows that

(12) $CP_1C' = (CGG'C')^{1/2}$.

3 - LIMITING FORM OF KALMAN-BUCY FILTER

In this section we study the limiting forms of Kalman-Bucy filter in two cases, that is, $G = 0$ and rank $(CGG'C') \geq 1$.

Case 1 $G = 0$

In this case, equations (11)and (12) become

(11a') $CP_0 = 0$

(11b') $AP_0 + P_0A' - P_1C'CP_1 = 0$

(11c') $AP_1 + P_1A' - P_2C'CP_1 - P_1C'CP_2 = 0$

(11d') $AP_2 + P_2A' - P_2C'CP_2 - P_1C'CP_3 - P_3C'CP_1 = 0$

(12') $CP_1C' = 0$

Let

$$X_c(A,B) = \sum_{i=0}^{n-1} A^i \text{ Range } B$$

(the controllable subspace of (A,B)),

$$X_{uo}(C,A) = \bigcap_{i=0}^{n-1} \mathrm{Ker}\; CA^i$$

(the unobservable subspace of (C,A))

then we have the following lemma.

LEMMA 1

$\mathrm{Ker}\; P_0 \supset X_c(A',C')$

(or Range $P_0 \subset X_{uo}(C,A)$)

$\mathrm{Ker}\; CP_1 \supset X_c(A',C')$

(or Range $P_1C' \subset X_{uo}(C,A)$)

Proof . We prove this lemma by induction.

Premultiplying (11b') by C and using (11a), (12'),

(13) $\quad CAP_0 = 0$

Suppose

(14) $\quad CA^iP_0 = 0$

(15) $\quad CA^iP_1C' = 0$

for i = 0, 1, 2, ..., k-1. Premultiplication of (11b') by CA^{k-1} yields

(16) $\quad CA^kP_0 + CA^{k-1}P_0A' - CA^{k-1}P_1C'CP_1 = 0.$

Then, it follows from equations (14), (15) and (16) that

(17) $\quad CA^kP_0 = 0.$

By pre- and postmultiplying (11b') by CA^k and $A'^k C'$, we have

(18) $$CA^{k+1}P_0A'^kC' + CA^kP_0A'^{k+1} - CA^kP_1C'CP_1A'^kC' = 0$$

Equations (17) and (18) give

$$CA^kP_1C' = 0$$

This completes the proof. □

<u>THEOREM 1</u>. Suppose the pair (C,A) is observable. Then

(a) $P_0 = P_1 = 0$,
(b) P_2 satisfies the ARE

(19) $$AP_2 + P_2A' - P_2C'CP_2 = 0.$$

Furthermore, if A has no imaginary axis eigenvalues, then there exists a solution P_2 of ARE (19) such that $A - P_2C'C$ is stable.

<u>Proof</u>. It follows from Lemma 1 that $P_0 = 0$

(20) $$CP_1 = 0,$$

since (C,A) is observable. By substituting (20) into (11b'), we have

(21) $$AP_1 + P_1A' = 0$$

Suppose now

(22) $$CA^iP_1 = 0 \text{ for } i = 0, 1, 2, \ldots, k-1.$$

Premultiplying (21) by CA^{k-1}

(23) $$CA^kP_1 + CA^{k-1}\,P_1' = 0$$

Equations (22) and (23) mean $CA^kP_1 = 0$. Therefore, $P_1 = 0$ by induction. Equation (19) follows from (11c) and (20).

The fact that $A - P_2C'C$ is stable is the well-known result of algebraic Riccati equation [7]. □

REMARK 1. The ARE (19) has more than one symmetric positive semidefinite solution which form a lattice with common maximum and common minimum element. The maximal solution is the only stabilizing solution. $P_2 = 0$ is a solution of the ARE (19). It is the maximum (or minimum) solution if $\mathrm{Re}\,\lambda(A) \le 0$ (or ≥ 0), where $\lambda(A)$ denotes an arbitrary eigenvalues of A. Otherwise $P_2 = 0$ is an intermediate solution [7].

COROLLARY 1. Suppose the pair (C,A) is observable and A has no imaginary axis. Then the limiting form of Kalman-Bucy filter (5) as ε tends to zero is

$$\frac{d}{dt}\hat{x}(t) = (A - P_2C'C)\,\hat{x}(t) + P_2C'\,y(t) + Bu(t) \tag{24}$$

where P_2 is a maximal solution of the ARE (19). Furthermore (24) is an identity observer (in the sense of Luenberger) for the system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \tag{25}$$

$$y(t) = Cx(t) \tag{26}$$

Proof. Equation (24) is a direct result of Theorem 1. Let $e(t) = \hat{x}(t) - x(t)$, then from (24), (25) and (26) we have

$$\frac{d}{dt}e(t) = (A - P_2C'C)\,e(t)$$

Therefore $e(t)$ tends to zero, since $A - P_2C'C$ is stable. This completes the proof. □

<u>Case 2.</u> $0 < \text{rank}\ (CGG'C') \leq r$.

From (11a), (11b) and (12), we have

(27) $$(CGG'C')^{1/2}CP_1 = C(AP_0 + GG')$$

Let

$$q = \text{rank}\ (CGG'C')$$

then there exists an orthonormal matrix U such that

(28a) $$CGG'C' = U'\ \Lambda^2\ U$$

(28b) $$\Lambda = \begin{array}{cc} \begin{bmatrix} \Lambda_q & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} q \\ r-q \end{array} \\ \begin{array}{cc} q \quad & \quad q \end{array} & \end{array}$$

(28c) $$\Lambda_q = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_q)\ \lambda_i > 0,\ i = 1, 2, \ldots, q$$

Suppose equation (27) has a solution (which may be not unique), then

(29) $$CP_1 = \Phi^+ C(AP_0 + GG') + Z$$

where Φ^+ is a pseudo-inverse of

(30) $$\Phi = (CGG'C')^{1/2}$$

and Z is an arbitrary r n matrix such that $\Phi Z = 0$ and $ZC' = 0$. Note that

(31) $$\Phi^+ = U'\ \Lambda^+\ U,$$

where

(32) $$\Lambda^+ = \begin{array}{cc} \begin{bmatrix} \Lambda_q^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} q \\ r-q \end{array} \\ \begin{array}{cc} q \quad & \quad r-q \end{array} & \end{array}$$

Then

$$\Phi^{+}Z = U'\Lambda^{+}UZ$$

(33) $$= U' [I_q \mid 0_{q\times(r-q)}]' \Lambda_q^{-1} [I_q \mid 0_{q\times(r-q)}]UZ$$

$$= 0$$

since $\Phi Z = 0$ implies $[I_q \mid 0_{q\times(r-q)}]\, UZ = 0$. Subsituting (29) into (11b) and using (33), we have

(34) $$(I_n - GG'C'\Phi^{+2}C)AP_0 + P_0A'(I_n - GG'C'\Phi^{+2})' - P_0A'C'\Phi^{+2}CAP_0 + \\ + GG' - GG'C'\Phi^{+2}CGG' - Z'Z = 0$$

Let us decompose UC into two blocks such that

(35a) $$UC = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

where

(35b) $$C_1 = [I_q \mid 0_{q\times(r-q)}]\, UC$$

(35c) $$C_2 = [0_{(r-q)\times q} \mid I_{(r-q)}]\, UC,$$

then, noting that

$$\Phi^{+2} = U'[I_q \mid 0_{q\times(r-q)}]' \Lambda_q^{-2} [I_q \mid 0_{q\times(r-q)}]U$$

the algebraic Riccati equation (34) becomes

$$(36a) \qquad (I_n - C_1^{\#}C_1)AP_0 + P_0A'(I_n - C_1^{\#}C_1)' - P_0A'C_1'\Lambda_q^{-2}C_1AP_0 + M(Z) = 0$$

where

$$(36b) \qquad C_1^{\#} = GG'C_1'\Lambda_q^{-2}$$

and

$$(36c) \qquad M(Z) = GG' - GG'C_1'\Lambda_q^{-2} C_1GG' - Z'Z$$

It can be easily shown from equations (28), (35) and (36b) that $C_1^{\#}$ is a right inverse of C_1, that is

$$(37) \qquad C_1C_1^{\#} = I_q$$

and

$$(38) \qquad C_2C_1^{\#} = 0.$$

Let S_1 be an n×q selector matrix which selects q independent columns from C_1, and let S_1^C be an n×(n-q) selector matrix which selects n-q columns of C_1 such that S_1 does not select. Define an n×n nonsingular matrix T_1 by

$$(39a) \qquad T_1 = \begin{bmatrix} C_1 \\ \\ V_1 \end{bmatrix}$$

with

$$(39b) \qquad V_1 = S_1^{C'}(I_n - C_1^{\#}C_1)$$

then a simple calculation yields

$$(40a) \qquad T_1^{-1} = [C_1^{\#} \; V_1^{\#}]$$

where

(40b) $\quad V_1^{\#} = S_1^C - S_1(C_1S_1)^{-1}\, C_1S_1^C$

Furthermore

(41) $\quad C_1T_1^{-1} = [I_q \mid 0_{q\times(n-q)}]$

(42) $\quad T_1C_1^{\#} = [I_q \mid 0_{q\times(n-q)}]'$

(43) $\quad T_1C_1^{\#}C_1T_1^{-1} = E_q$

(44) $\quad T_1(I_n - C_1^{\#}C_1)T_1^{-1} = I_n - E_q$

where

(45) $$E_q = \begin{array}{cc} \left[\begin{array}{cc} I_q & 0 \\ 0 & 0 \end{array}\right] & \begin{array}{c} q \\ n-q \end{array} \\ \begin{array}{cc} q & n-q \end{array} & \end{array}$$

Note that $CP_0 = 0$ if and only if

(46a) $\quad C_1P_0 = 0$

and

(46b) $\quad C_2P_0 = 0$

Note also that (27) has a solution (which may be not unique) if and only if

(47) $\quad C_2AP_0 = 0$

since $C_2G = 0$ from (28), (30), (31) and (35). From equations (39), (40) and (46a)

(48) $$T_1P_0T_1' = \begin{array}{cc} \left[\begin{array}{cc} 0 & 0 \\ 0 & \Pi_1 \end{array}\right] & \begin{array}{c} q \\ n-q \end{array} \\ \begin{array}{cc} q & n-q \end{array} & \end{array}$$

where

$$(49) \qquad \Pi_1 = V_1 P_0 V_1'$$

Equations (39a), (40b) and (49) yield

$$(50) \qquad P_0 = V_1^{\#} \Pi_1 V_1^{\#\prime}$$

Moreover, it follows from (46b), (47) and (50) that

$$(51) \qquad \begin{bmatrix} C_2 V_1^{\#} \\ \\ C_2 A V_1^{\#} \end{bmatrix} \Pi_1 = 0$$

Suppose now

$$(52) \qquad \text{rank} \begin{bmatrix} C_2 V_1^{\#} \\ \\ C_2 A V_1^{\#} \end{bmatrix} = \ell < n-q$$

Let S_2 be an $\ell \times 2(r-q)$ selector matrix which selects ℓ independent rows from $\begin{matrix} C_2 V_1^{\#} \\ C_2 A V_1^{\#} \end{matrix}$ and let S_3 be an $(n-q) \times \ell$ selector matrix which selects ℓ independent columns from

$$(53) \qquad \bar{C}_2 = S_2 \begin{bmatrix} C_2 V_1^{\#} \\ \\ C_2 A V_1^{\#} \end{bmatrix}$$

Define an $(n-q) \times (n-q)$ nonsingular matrix T_2 by

$$(54a) \qquad T_2 = \begin{bmatrix} \bar{C}_2 \\ V_2 \end{bmatrix}$$

with

$$(54b) \qquad V_2 = S_3^{C'} (I_{n-q} - \bar{C}_2^{*} \bar{C}_2)$$

where $\bar{C}_2^*$ is an arbitrary right inverse of $\bar{C}_2$ and S_3^C is the $(n-q)$ $(n-q-\ell)$ selector matrix which selects $(n-q-\ell)$ columns of $\bar{C}_2$ such that S_3 does not select. Then

(55a) $$T_2^{-1} = [\bar{C}_2^* \mid V_2^*]$$

where

(55b) $$V_2^* = S_3^C - S_3(\bar{C}_2S_3)^{-1}\bar{C}_2S_3^C$$

It follows from (51) and (54) that

(56) $$T_2\Pi_1T_2' = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ & \\ 0 & \Pi_2 \end{bmatrix} & \begin{array}{l} \ell \\ \\ n-q-\ell \end{array} \\ \begin{array}{cc} \ell & \quad n-q-\ell \end{array} & \end{array}$$

where

(57) $$\Pi_2 = V_2\Pi_1V_2'$$

By a simple calculation

(58) $$\Pi_1 = V_2^*\Pi_2V_2^{*\prime}$$

The above procedure gives us one of the main results of this paper.

<u>THEOREM 2</u>

(a) If $\ell = n-q$, then $P_0 = 0$
(b) If $\ell < n-q$, then

(59) $$P_0 = V_1^{\#}V_2^*\Pi_2V_2^{*\prime}V_1^{\#\prime}$$

where Π_2 is an $(n-q-\ell)\times(n-q-\ell)$ matrix solution of the algebraic Riccati equation

$$(60)\qquad \begin{aligned}&(V_2V_1AV_1^{\#}V_2^{*})\Pi_2 + \Pi_2(V_2V_1AV_1^{\#}V_2^{*})' - \Pi_2(C_1AV_1^{\#}V_2^{*})'\,\Lambda_q^{-2}(C_1AV_1^{\#}V_2^{*})\Pi_2\\ &\quad + V_2V_1M(Z)V_1'V_2' = 0\end{aligned}$$

Proof. (a) is obvious from (51). It follows from (39) and (40) that

$$(61)\qquad T_1(I_n - C_1^{\#}C_1)AT_1^{-1} = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ V_1AC_1^{\#} & V_1AV_1^{\#} \end{bmatrix} & \begin{array}{c} q \\ n-q \end{array} \\ \begin{array}{cc} q & \quad n-q \end{array} & \end{array}$$

By noting $C_1M(Z) = 0$, we have

$$(62)\qquad T_1M(Z)T_1' = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 0 & V_1M(Z)V_1' \end{bmatrix} & \begin{array}{c} q \\ n-q \end{array} \\ \begin{array}{cc} q & \quad n-q \end{array} & \end{array}$$

Pre- and postmultiplying (36a) by T_1 and T_1', and using (48), (61) and (62) we obtain

$$(63)\qquad (V_1AV_1^{\#})\Pi_1 + \Pi_1(V_1AV_1^{\#})' - \Pi_1(C_1AV_1^{\#})'\,\Lambda_q^{-1}(C_1AV_1^{\#})\Pi_1 + V_1M(Z)V_1' = 0$$

Substitution of (58) into (63) yields (60). Equation (59) follows from (50) and (58). □

REMARK 2. Theorem 2 is an extension of Theorem 1 in [3] to the case where CGG'C' may be not invertible.

REMARK 3. It is easy to prove that the pair $(C_1AV_1^{\#}V_2^{*},\ V_2V_1AV_1^{\#}V_2^{*})$ is observable if the pair (C_1,A) is observable.

REMARK 4. The condition $\ell = n-q$ implies $(n-r) < (r-q)$, i.e., rank $C_2 > n-r$.

REMARK 5. If $P_0 = 0$, then rank G must be less than or equal to r, since $P_1C'CP_1 = GG'$. Moreover, by a simple calculation , we have

(64) $$C_1P_1 = \Lambda_q^{-1} C_1GG'$$

(65) $$C_2P_1 = [0_{(r-q)\times q} \mid I_{(r-q)}] \, UZ$$

and

(66) $$GG' = GG'G' \Lambda_q^{-2} C_1GG'.$$

REMARK 6. Note that the solution of (60) depends on Z. Let $\Pi_2(Z)$ be a maximal solution of (60), then $\Pi_2(Z)$ is monotone with respect to M(Z), that is, $\Pi_2(Z_1) \geq \Pi_2(Z_2)$ if $M(Z_1) \geq M(Z_2)$ [1]. Therefore, $\Pi_2^0 \geq \Pi_2(Z)$ with $\Pi_2^0 = \Pi_2(0)$, since $M(0) \geq M(Z)$ for all Z. In the following we assume that Z = 0, since the maximum of the maximal solutions of (60) has an interest for us.

By capital letters $\hat{X}$, $\hat{U}$, and Y we mean the Laplace transforms of $\hat{x}$, $\hat{u}$ and y, respectively. Upon Laplace transformation, the Kalman-Bucy filter (5) becomes

(67) $$s\hat{X} = A\hat{X} + B\hat{U} + PC'\Gamma_\varepsilon$$

with

(68) $$\Gamma_\varepsilon = \frac{1}{\varepsilon^2}(Y - C\hat{X})$$

for $R = \varepsilon^2 I$. Then, by using (67),

$$\begin{aligned} s\varepsilon^2\Gamma_\varepsilon &= sY - Cs\hat{X} \\ &= sY - C(A\hat{X} + B\hat{U} + PC'\Gamma_\varepsilon) \end{aligned}$$

Therefore

(69) $$\Gamma_\varepsilon = (s\varepsilon^2 I_r + CPC')^{-1}(sY - CA\hat{X} - CB\hat{U}).$$

Premultiplying (69) by PC' and using the series expansion (10), we have

$$PC'\Gamma_{\varepsilon} = (P_1C' + \varepsilon P_2C' + 0(\varepsilon^2))\,[CP_1C' + \varepsilon(CP_2C' + sI) + 0(\varepsilon^2)]^{-1} \times (sY - CA\hat{X} - CB\hat{U}) \tag{70}$$

The following lemma will be useful.

LEMMA 2.

(a) $\lim_{\varepsilon\to 0} P_1C'\,[CP_1C' + \varepsilon(CP_2C' + sI) + 0(\varepsilon^2)]^{-1}$

$= P_1C'\Phi^{+}$

(b) $\lim_{\varepsilon\to 0} \varepsilon P_2C'\,[CP_1C' + \varepsilon(CP_2C' + sI) + 0(\varepsilon^2)]^{-1}$

$= 0$

Proof. Using (29) and (31) with Z = 0,

$$P_1C'\,[CP_1C' + \varepsilon(CP_2C' + sI) + 0(\varepsilon^2)]^{-1} = [GG' + P_0A']\bar{C}'\Lambda^{+}[\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1}\,U$$

where $\bar{C} = UC$.

Note that

$$\lim_{\varepsilon\to 0} \Lambda^{1/2}[\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1} = (\Lambda^{+})^{1/2},$$

since $\bar{C}P_2\bar{C}'$ is symmetric nonnegative definite. Therefore, (71) becomes

$$\lim_{\varepsilon \to 0} P_1 C'[CP_1C' + \varepsilon(CP_2C' + sI) + 0(\varepsilon^2)]^{-1}$$

$$= \lim_{\varepsilon \to 0} [GG' + P_0A'] \bar{C}'(\Lambda^+)^{3/2} \Lambda^{1/2} [\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1}U$$

$$= [GG' + P_0A'] \bar{C}(\Lambda^+)^2 U$$

$$= P_1 C\Phi^+$$

For the proof of (b) it is enough to show that

$$(73) \qquad \lim_{\varepsilon \to 0} \varepsilon P_2 \bar{C}' [\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1}U = 0$$

A simple calculation from (11c) yields that $\bar{C}P_2\bar{C}'$ is a solution of

$$\bar{C}AP_1\bar{C}' + \bar{C}P_1A'\bar{C}' = (\bar{C}P_1\bar{C}')\Lambda - \Lambda(\bar{C}P_2\bar{C}')$$

and

$$(74) \qquad \bar{C}P_2 = \Lambda^+ [\bar{C}AP_1 + \bar{C}P_1A' + \bar{C}P_2\bar{C}'\bar{C}P_1]$$

On the other hand

$$(75) \qquad \begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon\Lambda^+ [\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1} \\ &= \lim \varepsilon(\Lambda^+)^{3/2} \Lambda^{1/2}[\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1} \\ &= \lim_{\varepsilon \to 0} \varepsilon(\Lambda^+)^{3/2} + \lim_{\varepsilon \to 0} \Lambda^{1/2}[\Lambda + \varepsilon(\bar{C}P_2\bar{C}' + sI) + 0(\varepsilon^2)]^{-1} \\ &= 0 \end{aligned}$$

Equations (74) and (75) mean (73). This completes the proof. □

From (67), (70) and Lemma 2, we have

$$(76) \qquad s(\hat{X} - P_1C'\Phi^+Y) = (I_n - P_1C'\Phi^+C')(A\hat{X} + B\hat{U})$$

Let

$$W = X - P_1 C' \Phi^+ Y \tag{77}$$

then (76) becomes

$$sW = (I - P_1 C' \Phi^+ C')\,[A(W + P_1 C' \Phi^+ Y) + B\hat{U}] \tag{78}$$

Define another right inverse of C_1 by

$$C_1^* = C_1^{\#} + P_0 A' C_1 \Lambda_q^{-2} \tag{79}$$

Then, it follows from (29), (31) and (36b) that

$$\begin{aligned} P_1 C' \Phi^+ &= (GG' + P_0 A') \bar{C}' (\Lambda^+)^2 U \\ &= (GG' + P_0 A')\,[C_1' \Lambda_q^{-1} \mid 0_{n\times(r-q)}]U \\ &= [C_1^{\#} + P_0 A' C_1 \Lambda_q^{-2} \mid 0_{n\times(r-q)}]U \\ &= [C_1^* \mid 0_{n\times(r-q)}]U \end{aligned} \tag{80}$$

Equations (77), (78) and (80) yield

$$\hat{x}(t) = w(t) + [C_1^* \mid 0_{n\times(r-q)}]Uy(t) \tag{81}$$

and

$$\begin{aligned} \frac{d}{dt} w(t) &= (I_n - C_1^* C_1) AW(t) \\ &\quad + (I_n - C_1^* C_1)\, A[C_1^* \mid 0_{n\ (r-q)}]Uy(t) + (I_n - C_1^* C_1)\, Bu(t) \end{aligned} \tag{82}$$

Let us define an $n\times n$ nonsingular matrix T_3 by

$$T_3 = \begin{bmatrix} C_1 \\ V_3 \end{bmatrix} \tag{83a}$$

with

(83b) $$V_3 = S_1^c\,(I_n - C_1^* C_1)$$

Then, easily we have

(84) $$T_3^{-1} = [C_1^* \quad V_1^{\#}]$$

Furthermore

(85) $$T_3(I_n - C_1^* C_1)T_3^{-1} = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 0 & I_{(n-q)} \end{bmatrix} & \begin{array}{c} q \\ n-q \end{array} \\ \begin{array}{cc} q & \quad n-q \end{array} & \end{array}$$

Let

(86a) $$W_1 = [I_q \mid 0_{q\times(n-q)}]T_3 W$$

and

(86b) $$W_2 = [0_{(n-q)\times q} \mid I_{(n-q)}]T_3 W$$

Then, noting that

(87) $$T_3 A T_3^{-1} = \begin{bmatrix} C_1 A C_1^* & C_1 A V_1^{\#} \\ V_3 A C_1^* & V_3 A V_1^{\#} \end{bmatrix}$$

equations (82) becomes

(88) $$\frac{d}{dt} W_1(t) = 0$$

and

$$\frac{d}{dt} W_2(t) = V_3 A C_1^* W_1(t) + V_3 A V_1^{\#} W_2(t)$$

(89)
$$+ [V_3 A C_1^* \mid 0_{(n-q)\times(r-q)}] \, Uy(t)$$

$$+ V_3 B u(t)$$

By summarizing the above reuslts we have

<u>THEOREM 3</u>. The limiting form of Kalman-Bucy filter (5) as measurement noise tends to zero is as follows

(90)
$$\hat{x}(t) = V_1^{\#} W_2(t) + [C_1^* \mid 0_{n\times(r-q)}] Uy(t)$$

$$\frac{d}{dt} W_2(t) = (V_3 A V_1^{\#}) W_2(t)$$

(91)
$$+ [V_3 A C_1^* \mid 0_{(n-q)\times(r-q)}] Uy(t)$$

$$+ V_3 B u(t)$$

<u>Proof</u>. We may assume $W_1(o) = 0$, then $W_1(t) = 0$ from (88). Therefore, (81) and (89) become (90) and (91), respectively. □

<u>REMARK 7</u>. A simple calculation shows that

(92)
$$V_2 V_3 A V_1^{\#} V_2^* = V_2 V_1 A V_1^{\#} V_2^* - \Pi_2 (C_1 A V_1^{\#} V_2^*)' \, \Lambda_q^{-2} (C_1 A V_1^{\#} V_2^*)$$

where Π_2 is a maximal solution of (60), if $\ell < n-q$. Moreover, if the pair (C_1, A) is observable, then $V_2 V_3 A V_1^{\#} V_2^*$ is asymptotically stable, since the observability of (C_1, A) implies the observability of $(C_1 A V_1^{\#} V_2^*,\ V_2 V_1 A V_1^{\#} V_2^*)$ by Remark 3. We have also

(93)
$$V_3 A V_1^{\#} = V_1 A V_1^{\#} - \Pi_1 (C_1 A V_1^{\#})' \, \Lambda_q^{-2} (C_1 A V_1^{\#})$$

where Π_1 is given by (58) and is a maximal solution of (63) with constraints (51).

THEOREM 4. Suppose $V_3AV_1^{\#}$ is asymptotically stable, then the system (90) and (91) is an observer of the system

(94a) $$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

(94b) $$y(t) = Cx(t)$$

Proof. Recall that the p-dimensionnal system

(95a) $$\frac{d}{dt}\psi(t) = F\,\psi(t) + Ky(t) + SBu(t)$$

(95b) $$\hat{x}(t) = N_1\psi(t) + N_2y(t)$$

is an observer for the system (94) if and only if

(i) F is asymptotically stable
(ii) $SA - FS = KC$
(iii) $N_1S + N_2C = I_n$

[4], [5].

Let $p=n-q$, $F = V_3AV_1^{\#}$, $K = [V_3AC_1^{*-} \mid 0_{(n-q)\times(r-q)}]U$, $S = V_3$, $N_1 = V_1^{\#}$, and $N_2 = [C_1^* \mid 0_{n\times(r-q)}]U$. Then obviously condition (i) holds. Condition (ii) follows from the fact that

$$SA - FS = V_3A(I_n - V_1^{\#}V_3) = V_3AC_1^*C_1 = KC.$$

Also

$$N_1S + N_2C = V_1^{\#}V_3 + C_1^*C_1 = I_n$$

This completes the proof.

REMARK 8. Suppose $\ell < n-q$. Then by Remark 7, $V_2V_3AV_1^{\#}V_2^*$ is asymptotically stable, however, this fact doesn't always imply that $V_3AV_1^{\#}$ is asymptotically stable. A sufficient condition which guaranties the asymptotic stability of $V_3AV_1^{\#}$ is

(96a) $\bar{C}_2 V_1 A V_1^{\#} = 0$

and

(96b) $\bar{C}_2 V_1 A V_1^{\#} \bar{C}_2^{*}$ is asymptotically stable,

since

(97) $$T_2 V_3 A V_1^{\#} T_2^{-1} = \begin{bmatrix} \bar{C}_2 V_1 A V_1^{\#} \bar{C}_2^{*} & \bar{C}_2 V_1 A V_1^{\#} V_2^{*} \\ V_2 V_3 A V_1^{\#} \bar{C}_2^{*} & V_2 V_3 A V_1^{\#} V_2^{*} \end{bmatrix}$$

REMARK 9. Suppose $\ell = n-q$, then $C_1^{*} = C_1^{\#}$ and $V_3 = V_1$, since $P_0 = 0$. This case we have no guarantee that $V_3 A V_1^{\#}$ $(= V_1 A V_1^{\#})$ is asymptotically stable except in the case where $V_1 A V_1^{\#}$ itself is asymptotically stable.

REMARK 10. Theorem 4 is an extension of the results in [2] to the case where CGG'C' is singular.

4 - CONCLUSION

This paper has considered the limiting form of Kalman-Bucy filter as measurement noise tends to zero. By introducing the pseudo-inverse of CGG'C', we derived the $(n-q-\ell) \times (n-q-\ell)$ dimensional Riccati equation for $q = \text{rank } CGG'C'$ and $\ell = \text{rank} \begin{bmatrix} C_2 V_1^{\#} \\ C_2 A V_1^{\#} \end{bmatrix}$. It has also shown that, under suitable conditions, the limiting form corresponds to an observer of order n-q for deterministic systems.

Acknowledgment

I would like to thank sincerely Professor A. Bensoussan who suggested me the possibility of solving this problem.

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Imprimé en France

par

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